Zero-energy states for a class of quasi-exactly solvable rational potentials

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Abstract

Quasi-exactly solvable rational potentials with known zero-energy solutions of the Schroödinger equation are constructed by starting from exactly solvable potentials for which the Schrödinger equation admits an so(2,1) potential algebra. For some of them, the zero-energy wave function is shown to be normalizable and to describe a bound state.

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In quantum mechanics the number of exactly solvable (ES) potentials being limited, there have been recent searches for either quasi-exactly solvable(QES) [1] or conditionally exactly solvable (CES) [2, 3, 4, 5] systems, for which the energy spectrum is partly or completely known under certain constraint conditions among the potential parameters, respectively. Apart from being mathematically interesting, both types of potentials offer useful insights into the description of physical phenomena: for instance, one can establish a correspondence between QES problems and the spin-boson and spin-spin interacting models [6], while CES potentials may be shown to be related to the ES Coulomb, anharmonic, and Manning-Rosen potentials [5].

There has recently been a resurgence of interest [7] in tracking down potentials with zero binding energy. For one thing, an $E = 0$ quantal state is a QES system in its own right; for another, familiarity with the Coulomb problem persuades one to expect that all zero-energy states lie in the continuum and so are not normalizable. However, Daboul and Nieto [7] have made a systematic survey with radial power law potentials to arrive at several exceptional solutions, which include two cases where the $E = 0$ wave function is normalizable and the corresponding state is either bound or unbound. Bound $E = 0$ solutions have also been reported before. For example, within the framework of the general effective radial potentials for an interacting spin 1/2 particle, Barut [8] had pointed out long ago that exact zero-energy solutions could exist with properties of confinement (normalizable case) and leakage (non-normalizable case).

The purpose of this letter is to present another evidence of zero-energy normalizable solutions for a class of QES rational potentials. Our approach for constructing such a class combines elements of various methods for generating ES or CES potentials, including algebraic techniques based upon the use of so(2,1) as a potential algebra for the Schrödinger equation [9, 10, 11].

A few remarks on so(2,1) are in order. As shown by Wu and Alhassid [10], the underlying commutation relations of so(2,1), namely $[J_+, J_-] = -2J_0$, $[J_0, J_\pm] = \pm J_\pm$, may be given a differential realization $J_0 = -i\partial/\partial\phi$, and $J_\pm = e^{\pm i\phi}\left[\pm(\partial/\partial x) + F(x)\left(i\partial/\partial\phi \mp \frac{1}{2}\right) + G(x)\right]$, where the two functions F and G satisfy coupled differential equations

$$F' = 1 - F^2, \qquad \text{and} \qquad G' = -FG, \tag{1}$$

dashes denoting derivatives with respect to x. The Casimir operator $J^2 = J_0^2 \mp J_0 - J_\pm J_\mp$ being explicitly $J^2 = (\partial^2/\partial x^2) - F' \left[(\partial^2/\partial\phi^2) + \frac{1}{4} \right] + 2iG' (\partial/\partial\phi) - G^2 - \frac{1}{4}$, it follows readily that an irreducible representation of the potential algebra so(2,1) has basis states, which are eigenfunctions of different Hamiltonians, but conform to the same energy level. In other words, with $J_0|km\rangle = m|km\rangle$, and $J^2|km\rangle = k(k-1)|km\rangle$ $(m = k,\ k+1,\ k+2,\ \ldots)$, where $|km\rangle = \psi_{km}(x)e^{im\phi}$ are the basis functions, the coefficient functions $\psi_{km}(x)$ obey the Schrödinger equation

$$-\psi''_{km} + V_m\psi_{km} = -\left(k - \tfrac{1}{2}\right)^2 \psi_{km}, \tag{2}$$

where the one-parameter family of potentials is

$$V_m = \left(\tfrac{1}{4} - m^2\right) F' + 2mG' + G^2. \tag{3}$$

Wu and Alhassid [10] have considered particular solutions of (1) to deal with Morse, Pöschl-Teller, and Rosen-Morse potentials from (3). On the other hand, Englefield and Quesne [11] have explored more general possibilities to identify three classes of solutions from (1) according to whether $F^2 < 1$, $F^2 = 1$ or $F^2 > 1$:

$$\begin{array}{lll} \text{(I)} & F(x) = \tanh x, & G(x) = b\,\mathrm{sech}\,x, \\ \text{(II)} & F(x) = \pm 1, & G(x) = be^{\mp x}, \\ \text{(III)} & F(x) = \coth x, & G(x) = b\,\mathrm{cosech}\,x. \end{array} \tag{4}$$

Substitution of these solutions in (3) leads to non-singular Gendenshtein, Morse, and singular Gendenshtein potentials. These solutions encompass those obtained by Wu and Alhassid [10] in that Gendenshtein potentials are disguised versions of Pöschl-Teller potentials; further, for particular values of the parameters, one gets non-singular Rosen-Morse potentials from singular Gendenshtein potentials.

We now outline a procedure to obtain QES potentials from a very general scheme, which addresses the problem of generating ES potentials in quantum mechanics [12, 13].

Let us consider a change of variables $x \to f(u)$ resulting in the Schrödinger wave function transforming as $\psi(x) \to g(x)\chi(u(x))$. The standard form of the Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + [V(x) - E]\psi(x) = 0 \tag{5}$$

is thus modified to a more general expression

$$\frac{d^2\chi}{du^2} + Q(u)\frac{d\chi}{du} + R(u)\chi(u) = 0. \tag{6}$$

The functions $Q(u)$ and $R(u)$ are given by

$$Q(u) = \frac{u''}{u'^2} + \frac{2g'}{gu'}, \tag{7}$$

$$R(u) = \frac{g''}{gu'^2} + \frac{E - V(x)}{u'^2}, \tag{8}$$

where the dashes denote derivatives with respect to x.

From (7) and (8), the difference $E - V(x)$ may be represented in terms of the function u only:

$$E - V(x) = \frac{1}{2}\Delta V(u) + u'^2\left(R - \frac{1}{2}\frac{dQ}{du} - \frac{1}{4}Q^2\right), \tag{9}$$

where the quantity $\Delta V(u)$ is the so-called Schwartzian derivative [14],

$$\Delta V(u) = \frac{u'''}{u'} - \frac{3}{2}\left(\frac{u''}{u'}\right)^2. \tag{10}$$

It is clear from (9) that if the functions $Q(u)$ and $R(u)$ are known explicitly, we can try out various choices of $u(x)$ to arrive at an ES potential. In the literature, several studies [12, 13] of Eq. (9) have been made by comparing Eq. (5) with those differential equations whose solutions are analytically known.

We would like to point out that CES or QES potentials may also be obtained from the secondary differential equation (6) by putting $Q(u) = 0$. The latter implies $u'g^2 =$ constant, $\psi = (u')^{-1/2}\chi(u(x))$, and from (9) or (8)

$$E - V(x) = \tfrac{1}{2}\Delta V(u) + Ru'^2. \tag{11}$$

To get a meaningful representation of (11), we set $R = E_T - V_T(u)$, and use (10) along with the transformation $x = f(u)$ to get the result

$$E_T - V_T(u) = [E - V(f(u))](f'(u))^2 + \tfrac{1}{2}\Delta V(f(u)), \tag{12}$$

where

$$\Delta V(f(u)) = \frac{f'''(u)}{f'(u)} - \frac{3}{2}\left(\frac{f''(u)}{f'(u)}\right)^2. \tag{13}$$

In Eqs. (12) and (13), the dashes now stand for differentiation with respect to the variable u. Note that the above relation was also obtained by de Souza Dutra [2] by effecting a transformation $x = f(u)$ in the Schrödinger equation (5), and redefining the wave function as $\psi(x) = \sqrt{df(u(x))/du}\,\chi(u)$. However Eqs. (6) and (9) were first written down by Bhattacharjee and Sudarshan [12] in 1962, and contained, as we have just shown, de Souza Dutra's result as a particular case ($Q = 0$).

In recent papers [2, 3, 4, 5], the result (12) has been exploited by a number of authors to get CES potentials in the form $V(f(u))$ by choosing judiciously the transformation function $f(u)$, and assigning to V_T an ES potential with known energy spectrum and eigenfunctions.

Here we shall exploit it in a slightly different way to get QES potentials by taking so(2,1) as the potential algebra of the Schrödinger equation, and the resulting form of the potential V_m, given by (3), as V_T. We get in this way

$$\left(\tfrac{1}{4} - m^2\right)\left(1 - F^2\right) - 2mFG + G^2 - E_T = [f'(u)]^2[V(f(u)) - E] - \tfrac{1}{2}\Delta V(f(u)), \qquad (14)$$

where the derivatives of F and G have been removed by making use of Eq. (1).

Whereas $F(x)$ and $G(x)$ are known for the three cases summarized in (4), which are relevant for the potential algebra so(2,1), the mapping function $f(u)$ is unknown in (14). In the following, we propose to use a mapping function that transforms a half-line onto itself. It may be remarked that in Ref. [5], a form for $f(u)$ was employed which switched variable x to u transforming in the process the full line to the half-line.

Let us choose the mapping function to be

$$x = f(u) = (e^u - 1)^{-1}. \qquad (15)$$

It is obvious from (15) that $(0, \infty)$ is the domain of both variables x and u. Using (15), the Schwartzian derivative (13) reduces to $\Delta V(f(u)) = -1/2$.

Consider first the class I solutions obtained in (4). Substituting these into (14), and setting $E = 0$ give

$$\begin{aligned} V(f(u)) &= \left(1 - 4m^2 + 4b^2\right) \operatorname{sech} u(\operatorname{sech} u - 2) - 8mb(\tanh u \operatorname{sech} u - 2\tanh u + \sinh u) \\ &\quad - (4E_T + 1)\cosh u(\cosh u - 2) - 4\left(E_T + m^2 - b^2\right). \end{aligned} \qquad (16)$$

The expression (16) may be translated in terms of the variable x through the use of (15), leading to

$$\text{(I)} \qquad V(x) = -\frac{A}{\left(2x^2+2x+1\right)^2} - B\frac{2x+1}{x(x+1)\left(2x^2+2x+1\right)^2} + \frac{C}{x^2(x+1)^2}, \tag{17}$$

with

$$A = 4\left(m^2 - b^2\right) - 1, \qquad B = 4mb, \qquad C = -E_T - \tfrac{1}{4} = \left(k - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} = k(k-1). \tag{18}$$

This potential is our first example of QES potential with known $E = 0$ eigenvalue. For nonnegative values of B and C, its behaviour at the origin is similar to that of the (shifted) Coulomb effective potential $V_E(x) = \left(\hbar^2/2m\right)\left(l(l+1)/x^2\right) - \left(e^2/x\right)$. From (17), we indeed note that as $x \to 0$, the first term $\to -A$, the second term $\sim -B/x$, and the third term $\sim C/x^2$. On the other hand, for $x \to \infty$, $V(x) \to 0$.

To consider whether the $E = 0$ quantal state is normalizable, let us consider for instance the potential (17) for which $m = k$ (i.e., A, B, and C are connected by the relation $A = 1+4C+2\sqrt{1+4C}-B^2\left(1+\sqrt{1+4C}\right)^{-2}$). The corresponding wave function is $\psi_0(x) = \sqrt{f'(u(x))}\,\chi_0(u(x))$, where $\chi_0(u)$ is the ground state wave function of class I potential in Eq. (4), namely $\chi_0(u) \sim G^{k-\frac{1}{2}}\,h$ with $h = \exp\left[b\tanh^{-1}(\sinh u)\right]$ [11]. Since $\sqrt{f'(u(x))} \sim \sqrt{2x(x+1)}$, $\psi_0(x) \to 0$ as $x \to 0$, but $\psi_0(x) \sim x$ for $x \to \infty$. We therefore conclude that for $m = k$, the function ψ_0 is non-normalizable.

Turning now to the class II solutions of (4), we find (14) to yield another QES potential with $E = 0$,

$$\text{(II)} \qquad V(x) = \frac{A}{(x+1)^4} - \frac{B}{x(x+1)^3} + \frac{C}{x^2(x+1)^2}, \tag{19}$$

with

$$A = b^2, \qquad B = 2mb, \qquad C = -E_T - \tfrac{1}{4} = k(k-1). \tag{20}$$

Here, as $x \to \infty$, $V(x) \to 0$, while for $x \to 0$, $V(x) \to A - (B/x) + (C/x^2)$. Once again $V(x)$ mimics the Coulomb problem near the origin provided nonnegative values of B and C are considered. An analysis of the wave function corresponding to $E = 0$ for the potential with $m = k$ (i.e., with $BA^{-1/2} = 1 + \sqrt{1+4C}$) shows that $\psi_0(x) = \sqrt{f'(u(x))}\,\chi_0(u(x))$,

where $\chi_0(u) \sim G^{k-\frac{1}{2}}\, h$, and $h \sim \exp\left(-be^{-u}\right)$ [11]. It is easy to check that $\psi_0 \to \infty$ as $x \to \infty$, so that in this case too the wave function is non-normalizable.

Finally, we take up the class III solutions of (4). For the corresponding pair (F, G), and $E = 0$, $V(x)$ turns out to be

$$\text{(III)} \qquad V(x) = \frac{A}{(2x+1)^2} - \frac{B}{x(x+1)} + \frac{C}{x^2(x+1)^2}, \tag{21}$$

with

$$A = 4(m+b)^2 - 1, \qquad B = 4mb, \qquad C = -E_T - \tfrac{1}{4} = k(k-1). \tag{22}$$

This QES potential shares similar qualitative features with the previous ones as far as its behaviour as $x \to 0$ or $x \to \infty$ is concerned. We shall presently see that provided certain convergence condition is satisfied, the wave function of the $E = 0$ state turns out to be normalizable for the potentials for which $m = k$.

To enquire into the normalizability of the wave function, we note that for the class III solutions of (4), and $m = k$ [11],

$$\psi_0(x) \sim \sqrt{x(x+1)}[\operatorname{cosech} u(x)]^{k-\frac{1}{2}}[\tanh(\tfrac{1}{2}u(x)]^b \sim \frac{[x(x+1)]^k}{(2x+1)^{k+b-\frac{1}{2}}}. \tag{23}$$

Since we are dealing with the one-dimensional Schrödinger equation (5) on the half line, the normalization integral is given by

$$\int_0^\infty |\psi_0(x)|^2\, dx \sim \int_0^\infty \frac{[x(x+1)]^{2k}}{(2x+1)^{2k+2b-1}}\, dx. \tag{24}$$

To examine its convergence, we make use of the criterion that if $\lim_{x\to\infty} x^\alpha f(x) = L$, where $\alpha > 1$, then $\int_a^\infty f(x)\, dx$ converges absolutely; if $\lim_{x\to\infty} x^\alpha f(x) = L \neq 0$ and $\alpha \leq 1$, then $\int_a^\infty f(x)\, dx$ diverges. In the present case, $f(x) \sim x^{2k-2b+1}$ for $x \to \infty$. So if we take $\alpha = 2b-2k-1$, we have $\lim_{x\to\infty} x^\alpha f(x) = L \neq 0$ and the integral converges if $2b-2k-1 > 1$ or $b > k+1$, and diverges if $b \leq k+1$. Whenever the convergence condition $b > k+1$ is satisfied, the normalization integral (24) may be easily evaluated. For $2k \in \mathbb{N}$, one finds

$$\int_0^\infty |\psi_0(x)|^2\, dx \sim 2^{-4k-2} \sum_{m=0}^{2k} (-1)^m \binom{2k}{m} (b+m-k-1)^{-1}. \tag{25}$$

Similar results may be found for other values of k.

The convergence condition can also be expressed in terms of the parameters appearing in the potential (21). For $m = k + n$,

$$
\begin{aligned}
A &= (2k + 2b + 2n + 1)(2k + 2b + 2n - 1), && (26)\\
B &= 4(k + n)b, && (27)\\
C &= k(k - 1). && (28)
\end{aligned}
$$

These imply

$$
\begin{aligned}
k &= \tfrac{1}{2}\left(1 + \sqrt{1 + 4C}\right), && (29)\\
b &= \frac{B}{4n + 2 + 2\sqrt{1 + 4C}}, && (30)\\
A &= \left(2k + 2n + 1 + \frac{B}{2k + 2n}\right)\left(2k + 2n - 1 + \frac{B}{2k + 2n}\right)\\
&= \left(2n + 2 + \sqrt{1 + 4C} + \frac{B}{2n + 1 + \sqrt{1 + 4C}}\right)\\
&\quad \times \left(2n + \sqrt{1 + 4C} + \frac{B}{2n + 1 + \sqrt{1 + 4C}}\right). && (31)
\end{aligned}
$$

In Eq. (29), we selected the solution of Eq. (28) satisfying, for nonnegative values of C, the requirement imposed on so(2,1) unitary irreducible representation labels k, namely $k > 0$. We note that we actually obtain $k \geq 1$. For the case $m = k$ or $n = 0$, the convergence condition $b > k + 1$ then reads

$$
B > 4k(k + 1) = 4\left(1 + C + \sqrt{1 + 4C}\right), \tag{32}
$$

which in turn places non-trivial restriction on the run of A, as we shall now proceed to show.

The zeroes, as well as the regions of positivity and negativity, of potentials (21) are displayed in Table 1 in terms of A, B, and C, for $-\infty < A < \infty$, $0 \leq B < \infty$, and $0 \leq C < \infty$. Various types of potentials may be distinguished according to their behaviour over $(0, \infty)$, and/or the parameter range.

By using Eqs. (28), and (31) (for $n = 0$), expressing C and A in terms of k and B, it is now straightforward to show that those potentials (21) satisfying the convergence

condition (32) belong to either type I or type V of Table 1, according to whether $k > 1$ or $k = 1$ (equivalently $C > 0$ or $C = 0$). Examples for both types are displayed in Figs. 1 and 2, respectively.

It appears that contrary to the effective Coulomb potential, type I and V potentials approach zero from the top as $x \to \infty$. This explains why the corresponding $E = 0$ eigenstates are normalizable (and bound). Type I potentials behave as the effective Morse or spherical well potentials, which may provide phenomenological descriptions of alpha-decay, while type V potentials have the features of one of the effective radial power law potentials with $E = 0$ normalizable states found by Daboul and Nieto [7].

For simplicity's sake, the discussion of the $E = 0$ wave function normalizability has been carried out for those potentials for which $m = k$ or $n = 0$. It is clear however that its feasibility is by no way restrited to such values. For the class III potentials (21), for instance, the counterpart of Eq. (23) for $m = k + n$, and any $n \in \mathbb{N}$ can be found from the singular Gendenshtein potential wave functions given in Ref. [15]. It is given by

$$\psi_n(x) \sim \frac{[x(x+1)]^m}{(2x+1)^{m+b-\frac{1}{2}}} P_n^{(b-m,-b-m)} \left(\frac{2x^2+2x+1}{2x(x+1)} \right), \tag{33}$$

where m, b, and n are related to A, B, and C through (22), and $P_n^{(b-m,-b-m)}$ denotes a Jacobi polynomial.

In conclusion, we did provide some new examples of QES potentials with known $E = 0$ quantal states, and we did show that some of the latter may be normalizable and bound. It should be remarked that contrary to Daboul and Nieto, no example of normalizable, but unbound states has been found.

We constructed our QES potentials by a method inspired from general techniques for generating ES and CES potentials, and we used as starting point some ES potentials for which the Schrödinger equation admits an so(2,1) potential algebra. The method proposed here might also be useful for generating QES potentials from some other ES potentials.

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Figure captions

Figure 1: The potential $V(x)$ of Eq. (21) for $A = 399$, $B = 64$, $C = 2$, or $k = m = 2$, $b = 8$.

Figure 2: The potential $V(x)$ of Eq. (21) for $A = 63$, $B = 12$, $C = 0$, or $k = m = 1$, $b = 3$.

Table 1: Zeroes and sign of the potential $V(x)$, defined in Eq. (21), for $-\infty < A < \infty$, $0 \le B < \infty$, and $0 \le C < \infty$. The quantities x_{0+} and x_{0-} are defined by $x_{0\pm} \equiv \frac{1}{2}\left(-1+\sqrt{X_\pm/Y}\right)$, where $X_\pm \equiv A-2B-8C\pm2\sqrt{\Delta}$, $Y \equiv A-4B$, and $\Delta \equiv (B+4C)^2-4AC$.

Type	Parameters	Zeroes	Sign
I	$0 \neq 16C < 4B < A < \frac{(B+4C)^2}{4C}$	x_{0+}, x_{0-}	> 0 if $x < x_{0-}$ or $x > x_{0+}$
			< 0 if $x_{0-} < x < x_{0+}$
II	$0 \neq 16C < A = 4B$	$x_0 = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{B}{B-4C}}$	> 0 if $x < x_0$
			< 0 if $x > x_0$
III	$C \neq 0,\ A < \min\left(4B, \frac{(B+4C)^2}{4C}\right)$	x_{0-}	> 0 if $x < x_{0-}$
			< 0 if $x > x_{0-}$
IV	$0 \neq 4C < B,\ A = \frac{(B+4C)^2}{4C}$	$x_0 = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{B+4C}{B-4C}}$	> 0 if $x \neq x_0$
V	$C = 0,\ 0 \neq 4B < A$	$x_0 = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{A}{A-4B}}$	< 0 if $x < x_0$
			> 0 if $x > x_0$
VI	$A = 4B,\ 0 \neq 4C \ge B$	—	> 0
VII	$C \neq 0,\ 4B \neq A > \frac{(B+4C)^2}{4C}$	—	> 0
VIII	$0 \neq 4C > B,\ A = \frac{(B+4C)^2}{4C}$	—	> 0
IX	$B < 4C \neq 0,\ 4B < A < \frac{(B+4C)^2}{4C}$	—	> 0
X	$C = 0,\ A \le 4B \neq 0$	—	< 0
XI	$B = C = 0,\ A > 0$	—	> 0, $V(0) < \infty$
XII	$B = C = 0,\ A < 0$	—	< 0, $V(0) > -\infty$